# Gopakumar-Vafa Invariant and Perverse Sheaves (based on joint work with Jun Li)

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## Outline

# 1 Curve counting invariants

- $Y$ = smooth projective 3-fold over $\mathbb{C}$,
  $\pi_1(Y) = 0$, $h^{2,0}(Y) = 0$, $K_Y \cong \mathcal{O}_Y$.
- $\mathcal{O}_Y(1)$ = ample line bundle, $0 \neq \sigma \in H^{3,0}(Y)$.
- $g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(Y, \mathbb{Z})$.
- Expected: $\#\{$genus $g$ curves $C$ in $Y$ with $[C] = \beta\} < \infty$.

## Curve counting invariants of $Y$

- Expected: unchanged under deformation of $Y$.
- Defined as integrals on virtual fundamental classes of (compactified) moduli spaces of curves in $Y$ which admit perfect obstruction theories. Li-Tian, Behrend-Fantechi
- Different perspectives $\Rightarrow$ different compactifications $\Rightarrow$ different invariants.
- Gromov-Witten, Donaldson-Thomas, Pandharipande-Thomas, Fan-Jarvis-Ruan-Witten, etc

## Gromov-Witten invariant

- $C =$ reduced curve with at worst nodal singularity
- Kontsevich $f : C \to Y$ is stable if $\mathrm{Aut}(f) < \infty$.
- $\overline{M}_g(Y, \beta) =$ {stable maps to $Y$, $f_*[C] = \beta$}.
- $[\overline{M}_g(Y, \beta)]^{\mathrm{vir}} \in A_0(\overline{M}_g(Y, \beta))_{\mathbb{Q}} \xrightarrow{\#} A_0(pt)_{\mathbb{Q}} = \mathbb{Q}$.
- $N_g(\beta) = \#[\overline{M}_g(Y, \beta)]^{\mathrm{vir}} \in \mathbb{Q}$
- GW$\in \mathbb{Q}$ is not a pure curve counting:
  it comes with multiple cover contributions.

## Donaldson-Thomas invariant I

- $C \subset Y \Rightarrow I_C =$ ideal sheaf of $C$ in $Y$.
- $Hilb^P(Y) =$\{ideals $I$ in $\mathcal{O}_Y$ with $\chi(\mathcal{O}_Y/I \otimes \mathcal{O}_Y(m)) = P(m)$\}
  projective scheme with perf. obstr. th.: Thomas, Huybrechts.
- $[Hilb^P(Y)]^{\mathrm{vir}} \in A_0(Hilb^P(Y))$; DT invariant $=\#[Hilb^P(Y)]^{\mathrm{vir}}$.
- It comes with contributions from Hilbert scheme of points.
  Should divide out these contributions.
- Maulik-Nekrasov-Okounkov-Pandharipande conjecture GW $\sim$ DT.

## Donaldson-Thomas invariant II

- DT invariants are more generally defined for any compact moduli of stable sheaves. Thomas.

- A coherent sheaf $F$ on $Y$ is stable if $\frac{P(F')}{r(F')} < \frac{P(F)}{r(F)}$ for $0 \neq F' < F$. $M^P(Y) =$ q-proj moduli of stable sheaves on $Y$. Gieseker, Simpson. $[M^P(Y)]^{\mathrm{vir}} \in A_0(M^P(Y))$

- DT invariant $= \#[M^P(Y)]^{\mathrm{vir}} \in \mathbb{Z}$ if $M^P(Y)$ is compact.

- DT is not a pure curve counting unless $g = 0$.
  If $C$ is a smooth curve of genus $g > 0$ in $Y$ and $\deg P = 1$, $\{$line bundles on $C\}$ are contained in $M^P(Y)$ and contribute zero to DT.

## Other invariants

- stable pair = 1-dimensional sheaf + section with stability.
  Pandharipande-Thomas invariant $= \#[\{stable\ pairs\}]^{\mathrm{vir}}$
  Expected to be equivalent to GW. Bridgeland, Toda DT$\sim$PT.
- Chang-Li: stable maps to $\mathbb{P}^4$ with p-fields (2011). $\sim$ GW
- Fan-Jarvis-Ruan-Witten invariant = spin curve counting (c. 2007)
  AG theory by Chang-Li-Li (2013): cosection localization (K.-J.Li).
- All these invariants enumerate curves + extra data.
- Gopakumar-Vafa (BPS) invariant was proposed as a way of pure curve counting (1998).

## 2 Gopakumar-Vafa invariant

We imagine:

$M = \{(C, L) \,|\, C \subset Y, L \in \mathrm{Pic}(C), [C] = \beta\} \xrightarrow{for} S = \{C \,|\, C \subset Y\}$

Can think of $L$ as a 1-dim'l sheaf on $Y$.

Fiber over a smooth curve $C$ of genus $g$ is $\mathrm{Pic}^d(C)$.

$H^*(\mathrm{Pic}^d(C)) = H^*(E)^{\otimes g}$, $E=$ elliptic curve.

$H^*(E) = (\frac{1}{2}) \oplus 2(0)$ by hard Lefschetz.

$H^*(\mathrm{Pic}^d(C)) = \Big((\frac{1}{2}) \oplus 2(0)\Big)^{\otimes g} =: I_g$

Clebsch-Gordan rule: $(\frac{k}{2}) \otimes (\frac{l}{2}) = (\frac{k+l}{2}) \oplus \cdots \oplus (\frac{k-l}{2})$ for $k \geq l$

$\Rightarrow \{I_g\}$ form a basis for $Rep(sl_2)$.

We further imagine:

$\exists$ cohomology theory $\mathbb{H}$ with relative hard Lefschetz property:

$$M \xrightarrow{proj} S \xrightarrow{proj} pt, \quad \omega_L = c_1(\mathcal{O}_{M/S}(1)), \omega_R = c_1(\mathcal{O}_S(1)).$$

$\omega_L^k : \mathbb{H}^{-k}(M) \xrightarrow{\cong} \mathbb{H}^k(M)$, $\omega_R^k : \mathbb{H}^{-k}(S, -) \xrightarrow{\cong} \mathbb{H}^k(S, -)$.

We have an action of $(sl_2)_L \times (sl_2)_R$ on $\mathbb{H}(M)$.

$\mathbb{H}(M) = \oplus_{g \geq 0} I_g \otimes R_g$ by $(sl_2)_L$ action for some vector spaces $R_g$.

$(sl_2)_R$ acts on $R_g$ and can write $R_g = \oplus_{k \geq 0} (\frac{k}{2})^{\oplus m_k}$.

Define the trace (Euler number) of $(\frac{k}{2})$ as $\hat{\chi}(\frac{k}{2}) = (-1)^k(k+1)$.

GV invariant: $n_g(\beta) = \chi(R_g) = \sum(-1)^k(k+1)m_k$;

may be interpreted as the Euler number $e(\Omega_S)$ of the space of curves $S$.

## Mathematical theory?

Rigorous mathematical theory requires:

- $M =$ suitable moduli of 1-dimensional sheaves on $Y$;
  $S =$ suitable projective moduli of curves in $Y$;
  $h : M \to S$ projective morphism
- cohomology theory $\mathbb{H}$ with relative hard Lefschetz property.

Gopakumar-Vafa predicted $GV = \{n_g(\beta) \in \mathbb{Z}\} \Rightarrow \{N_g(\beta)\} = GW$:

$$\sum_{g,\beta} N_g(\beta) q^\beta \lambda^{2g-2} = \sum_{k,g,\beta} n_g(\beta) \frac{1}{k} \left(2\sin(\frac{k\lambda}{2})\right)^{2g-2} q^{k\beta}.$$

## 3 Proposals

S. Katz, Hosono-Saito-Takahashi, and others all agree:

$M =$ (seminormalization of) the moduli space of stable 1-dim'l sheaves $F$ on $Y$, $[F] = \beta$ and $\chi(F) = 1$.

$\mathfrak{hc} : M \to Chow^{1,\beta}(Y)$ Hilbert-Chow morphism.

$S =$ image of $M$ in $Chow^{1,\beta}(Y)$

S. Katz (c.2005) conjectured $n_0(\beta) = DT(M)$.

- Hosono-Saito-Takahashi (c.2001)
  proposed to use $IH^*(M)$ for the cohomology theory.

- M. Saito: $IH^*$ has relative hard Lefschetz property.

- $\chi(IH^*(M)) = \pm DT(M)$? No.

  Behrend (c.2005): $DT(M) = \chi(M, \nu_M) = \sum_k k \cdot \chi(\nu_M^{-1}(k))$.
  $\nu_M(x) = (-1)^{d-1}(\chi(Mil_f(x)) - 1)$ if $M = \mathrm{Crit}(f)$ for $f : V \to \mathbb{C}$
  $Mil_f(x) = f^{-1}(\delta) \cap B_\epsilon(x)$ for $0 < \delta << \epsilon << 1$.

  Ex. $y^2 = x^3$ has isolated critical point $\chi(IH) = 1$
  Milnor number 2 : $DT = 2$.

$P^\bullet = (P^a \to P^{a+1} \to \cdots \to P^{b-1} \to P^b)$: bdd cx of $\mathbb{Q}$-sheaves.
Kashiwara: $P^\bullet$ is a *perverse sheaf* if

1. $M = \sqcup M_\alpha$, $H^i(P^\bullet)|_{M_\alpha}$ is locally constant;
2. $\dim\{x \in X \,|\, \mathbb{H}^i(B_\varepsilon(x); P^\bullet) \neq 0\} \leq -i$ for all $i$;
3. $\dim\{x \in X \,|\, \mathbb{H}^i(B_\varepsilon(x), B_\varepsilon(x) - \{x\}; P^\bullet) \neq 0\} \leq i$ for all $i$.

Hypercohomology $\mathbb{H}^*(M, P^\bullet) = H^*(\Gamma(X, I^\bullet))$ where $P^\bullet \to I^\bullet$ is qis.

Beilinson-Bernstein-Deligne-Gabber: (i) $Perv(M)$ is an abelian category. (ii)$Perv(M)$ has gluing property: If $M = \cup U_\alpha$ open cover and $P^\bullet_\alpha \in Perv(U_\alpha)$ with gluing isomorphisms $\eta_{\alpha\beta} : P^\bullet_\alpha|_{U_{\alpha\beta}} \to P^\bullet_\beta|_{U_{\alpha\beta}}$ satisfying the cocycle condition, then $\exists\ P^\bullet \in Perv(M)$ such that $P^\bullet|_{U_\alpha} \cong P^\bullet_\alpha$.

M. Saito: If $P^\bullet$ underlies a semisimple polarizable Hodge module, then
i) $\mathbb{H}^*(M, P^\bullet)$ has relative hard Lefschetz property;
ii) $h : M \to S$ projective $\Rightarrow Rh_*P^\bullet =$ perverse sheaf underlying semisimple polarizable Hodge module (with shifts).

$\mathbb{H}^*(M, P^\bullet)$ has $sl_2 \times sl_2$-action from $M \xrightarrow{h} S \longrightarrow pt$.

## Perverse Sheaf of Vanishing Cycles

$f : V \to \mathbb{C}$ holomorphic map on cx manifold $V$ of dim $d$.
$X = \mathrm{Crit}(f) = \mathrm{zero}(df)$.

$P_f^\bullet := R\Gamma_{\{\mathrm{Re}\, f \le 0\}} \mathbb{Q}|_{f^{-1}(0)}[d] \in Perv(X)$
where $\Gamma_N I = \ker(I \to \imath_* \imath^* I)$ with $\imath : V - N \hookrightarrow V$.
$H^*(P_f^\bullet)|_x \cong \tilde{H}^{*+d-1}(Mil_f(x))$.
$\exists$ mixed Hodge module $M_f^\bullet$ with $\mathrm{rat}(M_f^\bullet) = P_f^\bullet$.

Joyce-Song (2008): $\forall x \in M$, $\exists$ open nbd $U$ of $x \in M$ such that
$U \cong \mathrm{Crit}(f)$ for $f : V \to \mathbb{C}$, $V$=cx mfd of dim $\dim T_x M$.

Corollary $\exists\ M = \cup_\alpha U_\alpha$ open cover and perverse sheaves $P^\bullet_\alpha \in Perv(U_\alpha)$ and mixed Hodge modues $M^\bullet_\alpha \in MHM(U_\alpha)$.

If $\exists$ gluing $P^\bullet$ of $P^\bullet_\alpha$, then $\chi(\mathbb{H}^*(M, P^\bullet)) = DT(M)$ by Behrend's result. (Katz conjecture)

If $\exists$ gluing $M^\bullet$ of MHMs $M^\bullet_\alpha \in MHM(U_\alpha)$, then $\hat{M}^\bullet = Gr^W(M^\bullet)$ is a semisimple polarizable Hodge module and $\hat{P}^\bullet = Gr^W(P^\bullet)$ is a perverse sheaf with $\mathrm{rat}(\hat{M}^\bullet) = \hat{P}^\bullet$.

The desired properties for GV theory
$\mathbb{H}^*(M, \hat{P}^\bullet)$ has $sl_2 \times sl_2$ action and $\chi(\mathbb{H}^*(M, \hat{P}^\bullet)) = DT(M)$ hold.

# 4 Categorification conjecture

Joyce-Song (2008): Can we glue $P^\bullet_\alpha \in Perv(U_\alpha)$ to a globally defined $P^\bullet \in Perv(M)$? The same for mixed Hodge modules?

K.-J.Li, Brav-Bussi-Dupont-Joyce-Szendroi (2012∼3): If $\exists$ universal family $\mathcal{E}$ on $M \times Y$ (recall $M$ is the moduli space of stable sheaves), then $\exists$ gluings $P^\bullet \in Perv(M)$ and $M^\bullet \in MHM(M)$.

The issue of cocycle condition uses an argument of Okounkov on the existence of a square root of $\det Ext(\mathcal{E}, \mathcal{E})$.

## Idea of proof

Seidel-Thomas twists: May assume that all sheaves are vector bundles.
$0 \to \mathcal{E}_1 \to H^0(\mathcal{E}(m)) \otimes \mathcal{O}(-m) \to \mathcal{E} \to 0$ for $m >> 0$.
Do the same to get $\mathcal{E}_2$, $\mathcal{E}_3$. Then $\mathcal{E}_3$ is a family of vector bundles.
$\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ have same deformation theories.

Donaldson-Thomas: holomorphic Chern-Simons theory.
Fix a hermitian complex vector bundle $E$.
$\mathcal{A} = \{\overline{\partial} : \Omega^0(E) \to \Omega^{0,1}(E) \,|\, \text{Leibniz}, \mathbb{C}\text{-linear}\} = \overline{\partial} + \Omega^{0,1}(\mathrm{End}\, E)$
$\overline{\partial}_a := \overline{\partial} + a \in \mathcal{A}$.

$\overline{\partial}_a$ is integrable iff $F^{0,2}(\overline{\partial}_a) = \overline{\partial}a + a \wedge a = 0$.
Holomorphic CS functional

$$CS(a) = \frac{1}{8\pi^2} \int_Y (a \wedge \overline{\partial}a + \frac{2}{3} a \wedge a \wedge a) \wedge \sigma.$$

$\mathrm{Crit}(CS) = \mathrm{zero}(F^{0,2}(\overline{\partial}_a))$=$\{$holomorphic structures on $E\}$
$M = \mathrm{Crit}(CS)/\mathcal{G} \subset \mathcal{A}/\mathcal{G} = \mathcal{B}$.

Joyce-Song: $V = \{\overline{\partial}_a \,|\, \overline{\partial}^* a = 0 = \overline{\partial}^* F^{0,2}(\overline{\partial}_a), |a| < \epsilon\} \subset \mathcal{A}$
cx mfd of dim $\dim T_x M$.
$f = CS|_V : V \to \mathbb{C}$, $\mathrm{Crit}(f) = V \cap \mathrm{Crit}(CS)$. Joyce-Song chart.

$\overline{\partial}^* a = 0 = \overline{\partial}^* F^{0,2}(\overline{\partial}_a) \Leftrightarrow \overline{\partial}\overline{\partial}^* a = 0$, $\overline{\partial}^*(\overline{\partial} a + a \wedge a) = 0$
$\Leftrightarrow L_{\overline{\partial}}(a) = (\overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}) a + \overline{\partial}^*(a \wedge a) = 0$. Elliptic op.

Can find a subspace $\Xi \subset \Omega^{0,1}(\mathrm{End}\, E)$ such that $\ker \Delta_{\overline{\partial}} \subset \Xi$ and $CS_2$ is nondegenerate on $\Xi/\ker\Delta_{\overline{\partial}}$. Define

$$V_\Xi = \{\overline{\partial}_a \mid L_{\overline{\partial}}(a) \in \Xi, |a| < \epsilon\}, \quad f = CS|_{V_\Xi}$$

hol ftn on cx mfd of dim $\dim \Xi$. We call $(V, f)$ a CS chart.

Continuous family of CS charts $\mathcal{V}_\alpha$ (with local triviality) on open $U_\alpha$ and homotopy of CS charts on $U_{\alpha\beta}$ give gluing isoms $\eta_{\alpha\beta}$ : $P^\bullet_\alpha|_{U_{\alpha\beta}} \to P^\bullet_\beta|_{U_{\alpha\beta}}$.

Cocycle condition $\Leftrightarrow$ existence of square root of $\det Ext(\mathcal{E}, \mathcal{E})$.

## Gopakumar-Vafa invariant

- $P^\bullet =$ perverse sheaf on $M$ underlying a MHM.
- $\hat{P}^\bullet = \mathrm{Gr}^W P^\bullet$ direct sum of the graded parts by the weight filtration
  $\Rightarrow \mathbb{H}^*(M, \hat{P}^\bullet)$ has hard Lefschetz and $sl_2 \times sl_2$ action
  $\Rightarrow$ GV (BPS) invariant $n_g(\beta)$. [K.-J.Li]
- K.-J.Li $n_0(\beta) = DT(M)$

- Hosono-Saito-Takahashi

$$\sum_{g,\beta} N_g(\beta) q^\beta \lambda^{2g-2} = \sum_{k,g,\beta} n_g(\beta) \frac{1}{k} \left(2\sin(\frac{k\lambda}{2})\right)^{2g-2} q^{k\beta}$$

holds for elliptic K3 fibered Calabi-Yau 3-folds, CY3 in Weierstrass model $\pi : Y \to S$ over del Pezzo surface with elliptic general fiber $F$.

- Thomas: Checked our GV theory is compatible with conjectural Katz-Klemm-Pandharipande formula (2014) for motivic BPS invariant (Choi-Katz-Klemm, 2012) .

- Conjectural wall crossing formula

$$n_g^+(\beta) - n_g^-(\beta) = (-1)^{\chi-1}\chi \cdot \sum_{h=0}^{g} n_h(\beta_1)n_{g-h}(\beta_2)$$

for $\beta = \beta_1 + \beta_2$, $\chi = \chi(E_1, E_2)$.

Thank you for your attention!